GRAVITATING SINE WAVES, GENERAL RELATIVITY, AND THE GOLDEN RATIO

RUSS HANUSH

Abstract. A link between general relativity and the golden ratio is investigated in a coordinate system of time, space, and mass axes. The golden ratio gives the arithmetic mean between the extremes of mass in the mass-space plane as the value of a parameter γ that relates the dilation of a sine wave in the space-time plane to the acceleration of gravity a in the mass-space plane by $a = 1 - \gamma^3$. When $\gamma = \varphi^n$, lines of constant γ have slopes of $\frac{1}{2}\varphi^n L_n$ for all odd n, and $\frac{1}{2}\varphi^n F_n\sqrt{5}$ for all even n. When $n = 1$, The intersection of that line and $a = 1 - \gamma^3$ also occurs at the point given by $\gamma = \varphi$, the acceleration of gravity at that same point is $2\varphi^2$ changing at a rate of φ; the sine wave in the space-time plane has a period of $\frac{2\pi}{\varphi}$.

1. Introduction

Consider a sine wave in the space-time plane with the equation

$$p(t) = \gamma \sin \gamma t. \tag{1.1}$$

For $0 < \gamma \leq 1$ this sine wave has an amplitude of γ, a period of $\frac{2\pi}{\gamma}$, and, with respect to time, a first derivative of

$$p'(t) = \gamma^2 \cos \gamma t, \tag{1.2}$$

a second derivative of

$$p''(t) = -\gamma^3 \sin \gamma t, \tag{1.3}$$

and *always* encloses an area of 4 as bounded by the t-axis. The invariance of this area is a fundamental property of space-time Pierseaux [9] describes as *Poincaré's area invariant.* It is the geometric interpretation of the invariance of the speed of light, and is why γ can never equal zero.

In special relativity, γ is given in the space-time plane by

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}} \tag{1.4}$$

where v is the observed velocity and c is the speed of light [3], [5, ch. XII], and our interest would lie in the first derivative (1.2). Indeed, Jozsef [7] has noted the golden ratio $\varphi = \frac{\sqrt{5}-1}{2}$ as the Lorentz transformation $\frac{1-\frac{v}{c}}{\gamma}$ in this plane when $v = \frac{1}{\sqrt{5}}c$. In fact, it should also be noted that when

$$\frac{v^2}{c^2} = \varphi, \qquad \gamma = \varphi, \tag{1.5}$$

$$\frac{v^2}{c^2} = \varphi^2, \qquad \gamma = \sqrt{\varphi}, \tag{1.6}$$

and

$$\frac{v^2}{c^2} = \varphi^3, \qquad \gamma = \varphi\sqrt{2}. \tag{1.7}$$

However, the focus of this paper is on the golden ratio in *general* relativity. For that we investigate a mass-space plane. In the mass-space plane γ is given by

$$\gamma = \sqrt{1 - 2\frac{Gm}{c^2 r}}, \tag{1.8}$$

where G is the gravitational constant, m is a mass, and r is the radius of that mass [4], [5, app. III (c)], and our interest lies in the second derivative (1.3). For the purpose of this discussion, these time, space, and mass axes are scaled to Natural units [11] with one temporal unit equal to 1 second, one spatial unit equaling 3×10^8 meters, and one mass unit equal to 4×10^{35} kg. In Natural units certain physical constants such as, c, G, and $\hbar$ are set equal to 1 allowing the suppression of those constants in the equations for this space, i.e., $E = mc^2$ becomes $E = m$, and (1.8) becomes $\gamma = \sqrt{1 - 2\frac{m}{r}}$; the suppressed constants may be inserted back into the equations at any time for dimensional analysis or emphasis.

2. Gravitating Sine Waves

In the space-time plane, (1.1) and its infinite derivatives represent every possible order of motion, and if one doesn't get too tied down to the concept of linear time, these waves (figure 1) represent every possible kinematic state a particle can take on independent of the temporal order in which those states occur. In this space the maximum speed possible is c, a slope of 1, and clearly, since no more speed can be gained, the magnitude of the acceleration must be exactly zero at that point and can only increase from there. The minimum speed is zero, and respectively, at that same point on the curve, the maximum acceleration is attained for the longest possible time since velocity is furthest from light speed; the second derivative tells us that the maximum acceleration is c/sec.

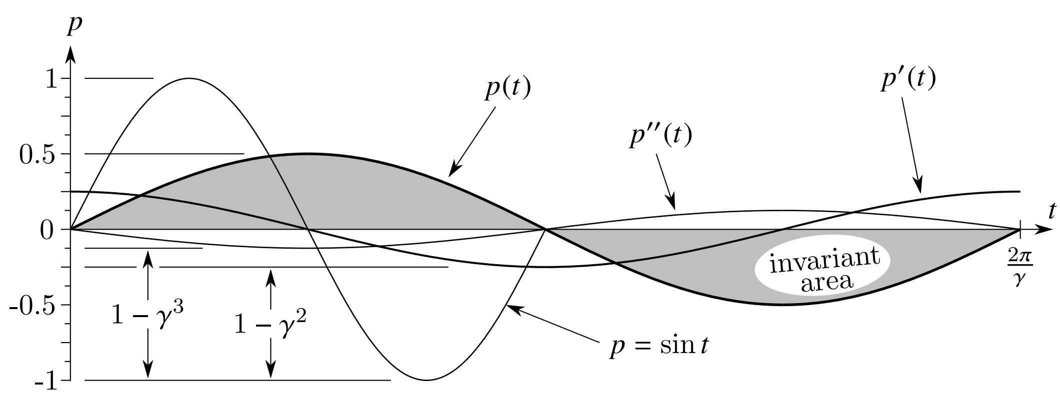

Figure 1. The space-time plane with $p(t) = \gamma \sin(\gamma t)$, $p'(t)$, and $p''(t)$ graphed on the same axes with period $\frac{2\pi}{\gamma}$, $\gamma = \frac{1}{2}$ is shown; $p = \sin t$, i.e, $\gamma = 1$, is shown for comparison. The differences $1 - \gamma^2$ and $1 - \gamma^3$, give v^2 and a, respectively. The invariant area is shaded.

As γ decreases the curve flattens out, and from the first derivative (1.2) we find that at its maximum amplitude the velocity is simply γ^2. This can be significantly less than c and limits the range over which an object can change its motion. This reduction would be quantified as $1 - \gamma^2$, for which an expression can be derived by rearranging (1.4) for $1 - \gamma^2$:

$$v^2 = 1 - \gamma^2. \tag{2.1}$$

From the second derivative (1.3) we find that at its maximum amplitude the acceleration is simply γ^3. But what changes that acceleration? That requires a force, which when applied uses energy which must be conserved. Therefore, the change in acceleration must be accounted for. If the range of velocities, as previously discussed, and accelerations is decreased from some universal maximum, i.e., the speed of light, then the object's ability to change its motion has been decreased. To an outside observer this would appear to be a resistance to change in motion: the definition of inertia, a.k.a., mass. For motion to be altered, an external force must be applied, and since $F = ma$, the gravitational acceleration a of the mass involved should be proportionate to the change in the amplitude of the second derivative

$$a \propto 1 - \gamma^3. \tag{2.2}$$

To determine this proportionality it is necessary to consider general relativity in the mass-space plane.

3. General Relativity

This factor γ that quantifies the dilation of the sine wave is proportionate to $\frac{m}{r}$, while gravitational acceleration is proportionate to $\frac{m}{r^2}$; two objects can have the same γ but vastly different gravitational accelerations. If we graph this relation in the mass-space plane by solving (1.8) for m as a function of r, and substituting radius for position on the space axis, γ becomes a parameter describing lines of constant slope where

$$m = \frac{1}{2}(1 - \gamma^2)r, \tag{3.1}$$

as shown in figure 2.

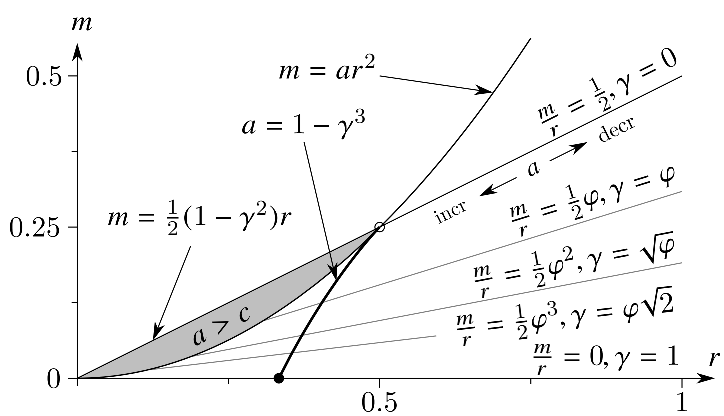

Figure 2. The mass-space plane with lines of constant γ having slopes of $\frac{1}{2}\varphi^n$ shown; the direction of increasing and decreasing gravitational acceleration is indicated. The parabola $m = ar^2$ designates the lower boundary of the shaded area representing accelerations greater than c/s. The curve $a = 1 - \gamma^3$ that relates general relativity in the mass-space plane to the sine wave in the space-time plane is shown with its excluded endpoint where the invariance of area under the sine wave would be violated: the transition to $m = ar^2$ is smooth and continuous.

The maximum slope approaches $\frac{1}{2}$, but as discussed earlier, a γ of zero, equal to a slope of $\frac{1}{2}$, violates the invariance of area under the curve. The minimum slope is zero for the condition

$m = 0$, $(\gamma = 1)$. Relations of the golden ratio similar to (1.5), (1.6), and (1.7) also show up in this space. Substituting $2\frac{m}{r}$ for v^2 in (1.4), we find that when

$$2\frac{m}{r} = \varphi, \qquad \gamma = \varphi, \tag{3.2}$$

$$2\frac{m}{r} = \varphi^2, \qquad \gamma = \sqrt{\varphi}, \tag{3.3}$$

and

$$2\frac{m}{r} = \varphi^3, \qquad \gamma = \varphi\sqrt{2},. \tag{3.4}$$

And since the slope in the mass-space plane is a function of γ, when

$$\gamma = \varphi, \qquad \frac{m}{r} = \frac{1}{2}\varphi, \tag{3.5}$$

$$\gamma = \sqrt{\varphi}, \qquad \frac{m}{r} = \frac{1}{2}\varphi^2 \tag{3.6}$$

and

$$\gamma = \varphi\sqrt{2}, \qquad \frac{m}{r} = \frac{1}{2}\varphi^3. \tag{3.7}$$

Moving along any line of constant γ changes the respective gravitational acceleration: toward the origin increases a, away from the origin decreases a. The parabola $m = ar^2$, for $a = 1$, has also been added to figure 2 to designate the boundary condition of accelerations greater than c/s.

For each line of constant γ there is exactly one point (r_γ, m_γ) such that

$$a = 1 - \gamma^3, \tag{3.8}$$

where a is the Newtonian acceleration given by

$$a = \frac{m}{r^2}. \tag{3.9}$$

And since γ is continuous for $0 < \gamma \leq 1$, these points form a continuous parametric curve. The parametric equations for r_γ and m_γ are:

$$r_\gamma = \frac{1}{2}\frac{(1-\gamma^2)}{(1-\gamma^3)} \tag{3.10}$$

and

$$m_\gamma = \frac{1}{4}\frac{(1-\gamma^2)^2}{(1-\gamma^3)}. \tag{3.11}$$

The expression for r_γ is derived by expressing a as $(m_\gamma/r_\gamma)/r_\gamma$, and setting it equal to the predicted condition, (3.8) such that

$$\frac{\frac{m_\gamma}{r_\gamma}}{r_\gamma} = 1 - \gamma^3. \tag{3.12}$$

Rearranging (1.8) for $\frac{m}{r}$ gives

$$\frac{m_\gamma}{r_\gamma} = \frac{1}{2}(1-\gamma^2) \tag{3.13}$$

at the expected conditions. Substituting (3.13) for the numerator on the left side of (3.12) and rearranging for r_γ gives (3.10); the expression for m_γ is derived by substituting (3.10) for r in (3.1).

From these parametric equations, an equation for m_γ as a function of r_γ can be derived. Taking (3.10) and multiplying both sides by $2(1-\gamma^3)$, then subtracting $(1-\gamma^2)$ and expanding gives a third-degree polynomial equal to zero:

$$-2r_\gamma\gamma^3 + \gamma^2 - (1 - 2r_\gamma) = 0.$$

Upon examination it can be seen that $\gamma = 1$ is a solution. Dividing that factor out gives the second-order depressed polynomial

$$-2r_\gamma\gamma^2 + (1 - 2r_\gamma)\gamma + (1 - 2r_\gamma) = 0$$

which can now be solved for γ using the quadratic formula so that we find

$$\gamma = \frac{-(1 - 2r_\gamma) \pm \sqrt{(1 - 2r_\gamma)^2 - 4(-2r_\gamma)(1 - 2r_\gamma)}}{2(-2r_\gamma)}.$$

Factoring out a $(1 - 2r_\gamma)$ in the discriminant and simplifying gives

$$\gamma = \frac{-(1 - 2r_\gamma) \pm \sqrt{(1 - 2r_\gamma)(6r_\gamma + 1)}}{-4r_\gamma}.$$

Dropping the extraneous root and substituting this expression for γ in (3.1) gives

$$m_\gamma = \frac{1}{2}\left[1 - \left(\frac{-(1 - 2r_\gamma) - \sqrt{(1 - 2r_\gamma)(6r_\gamma + 1)}}{-4r_\gamma}\right)^2\right]r_\gamma.$$

Once expanded and simplified, our expression for m_γ is

$$m_\gamma = \frac{12r_\gamma^2 - \sqrt{(1 - 2r_\gamma)^3(6r_\gamma + 1)} - 1}{16r_\gamma}; \tag{3.14}$$

physically constrained to $\frac{1}{3} \le r_\gamma \le \frac{1}{2}$, $0 \le m_\gamma \le \frac{1}{4}$, and $0 < \gamma \le 1$. This curve is smooth and continuous with $m = ar^2$ at $a = 1$ and $r = \frac{1}{2}$, with a derivative equal to 1 as $\lim_{r\to 1/2^-}$.

Each line of constant γ intersects this curve at the point (r_γ, m_γ). For that line, at that point, (3.8) holds true. For other gravitational accelerations along that same line of constant γ, the relation

$$ar = r_\gamma(1 - \gamma^3) \tag{3.15}$$

holds true. Einstein used the relation of tangential velocity to centripetal acceleration

$$v_t^2 = a_c r, \tag{3.16}$$

and his Principal of Equivalence of centripetal and gravitational accelerations, using (3.9) to make the leap from (1.4) to (1.8) (after natural units are employed) [4], [5, app. III (c)]. Here, replacing a in (3.16) with $\frac{m}{r^2}$, we find that

$$ar = \frac{m}{r}, \tag{3.17}$$

and substituting that into (1.8) gives

$$\gamma = \sqrt{1 - 2ar}. \tag{3.18}$$

Solving for ar gives $\frac{1}{2}(1 - \gamma^2)$, which is recognizable from our expression for r_γ (3.10). Substituting ar for $\frac{1}{2}(1 - \gamma^2)$ in (3.10) and rearranging gives (3.15), and as such any gravitational acceleration can be determined from the sine wave (1.1) by the relation

$$a = \frac{r_\gamma}{r}(1 - \gamma^3). \tag{3.19}$$

4. The Golden Ratio

Having established the gravitation of (1.1), we turn our investigation toward its relationship to φ and general relativity in this dimensional space. We begin with some preliminary findings. Given that the constraint for $\{\gamma \mid 0 < \gamma \leq 1\}$ is purely physical, it bears mention that φ splits γ in golden section. From a purely mathematical perspective, any function of a form similar to γ, such as $f(x) = \sqrt{1 - x^n}, n = 0, 1, 2, \ldots$, should be defined when $-\infty < x \leq 1$ for all odd n, and $-1 \leq x \leq 1$ for all even n, therefore, this section is in the golden ratio only in the physical domain.

Turning to the space-time plane we first note a relation of π to φ: the period of (1.1) when $\gamma = \varphi^n$ is $\frac{2\pi}{\varphi^n}$. However, in order to work with higher powers of φ in this space, we must use properties of the golden ratio to simplify their form. From eqs. (2.1), (3.10), and (3.11), it is noticed that factors of the form $1 - \gamma^2$ play a significant role in the equations for this space, therefore simplified forms of $1 - \varphi^{2n}$ would be useful in studying the case where $\gamma = \varphi^n$. A reduction formula can be derived by factoring out a φ^n, so that we have a form similar to Vajda's (58) [10] on the right side of

$$1 - \varphi^{2n} = \varphi^n(\varphi^{-n} - \varphi^n). \tag{4.1}$$

The factor $(\varphi^{-n} - \varphi^n)$ reduces to an alternating sequence of Lucas and Fibonacci numbers: $L_1, F_2\sqrt{5}, L_3, F_4\sqrt{5}, \ldots$, where $L_0 = 2$, and $F_0 = 0$, that Vajda cites as Lamé's sequence, which also has the property that ratios of sequential terms converge to φ^{-1}; as such, ratios of sequential terms of the right side of (4.1) converge to 1. From this we get two reduction formulas: for $n = 1, 3, 5, \ldots$

$$1 - \varphi^{2n} = \varphi^n L_n, \tag{4.2}$$

and for $n = 2, 4, 6, \ldots$

$$1 - \varphi^{2n} = \varphi^n F_n\sqrt{5}. \tag{4.3}$$

When $\gamma = \varphi^n$, v^2 is given by (4.2) for all odd n, and (4.3) for all even n. When φ^n is substituted for v^2 or v in (1.4), no related pattern is apparent. Acceleration in the space-time plane is the same as acceleration in the mass-space plane, therefore, we turn to (3.8) and find that we need factors containing φ^{3n}. This introduces odd powers of φ in (3.8), and as such only $n = 1$, and even powers of φ resolve using this method: $n = 1$ is expressible by the identity $1 - \varphi^3 = 2\varphi^2$, but the odd powers become inexpressible as simple powers of φ with factors of L_n or F_n. However, for the first two powers of φ, both positive and negative, some interesting values are noted in Table 1, along with values for some of the other quantities discussed here.

γ	r_γ	m_γ	$\frac{m}{r}$	$a = \frac{m}{r^2}$	$\frac{d}{dr}$(3.14)	$\frac{d}{dr}ar_\gamma^2$	v^2
φ^{-2}	$\frac{\sqrt{5}}{8}\varphi$	$-\frac{5}{16}\varphi^{-1}$	$-\frac{\sqrt{5}}{2}\varphi^{-2}$	$-4\varphi^{-3}$	—	$-\sqrt{5}\varphi^{-2}$	$-\sqrt{5}\varphi^{-2}$
φ^{-1}	$\frac{1}{4}$	$-\frac{1}{8}\varphi^{-1}$	$-\frac{1}{2}\varphi^{-1}$	$-2\varphi^{-1}$	—	$-\varphi^{-1}$	$-\varphi^{-1}$
φ	$\frac{1}{4}\varphi^{-1}$	$\frac{1}{8}$	$\frac{1}{2}\varphi$	$2\varphi^2$	$\frac{5}{2}\varphi$	φ	φ
φ^2	$\frac{\sqrt{5}}{8}\varphi^{-1}$	$\frac{5}{16}\varphi$	$\frac{\sqrt{5}}{2}\varphi^2$	$4\varphi^3$	—	$\sqrt{5}\varphi^2$	$\sqrt{5}\varphi^2$

Table 1. Values of r_γ and m_γ on the curve $a = 1 - \gamma^3$, the slopes of lines of constant γ, acceleration in both the space-time and mass-space planes, the slopes of lines tangent to the curves $a = 1 - \gamma^3$ and $m = ar^2$, respectively, and v^2 in the space-time plane are shown for φ^n when $n = \pm 1, \pm 2$.

In the mass-space plane we first note that the slope $\frac{m}{r}$ of lines of constant γ in this plane is given by (3.13). Then, when $\gamma = \varphi^n$ the slope of these lines is of the form

$$\frac{m}{r} = \frac{1}{2}\varphi^n L_n \tag{4.4}$$

for $n = 1, 3, 5, \ldots$, and

$$\frac{m}{r} = \frac{1}{2}\varphi^n F_n\sqrt{5} \tag{4.5}$$

for $n = 2, 4, 6, \ldots$. The equations for r_γ and m_γ themselves contain a factor of the form $1 - \gamma^3$, which is canceled in (4.4) and (4.5), and as such quickly become inexpressible in our terms.

Upon graphing the location of (r_γ, m_γ) for $\gamma = \varphi$ in the mass-space plane (figure 3) it quickly becomes apparent that φ occupies a special place in this space. A prediction in the space-time plane, about the second derivative of a sine wave and $1 - \gamma^3$, results in a curve in the mass-space plane given by a parameter that when set equal to φ gives the arithmetic mean between the extremes of mass on that curve; the curve that relates gravitational accelerations and $1 - \gamma^3$, at the point that the line of constant $\gamma = \varphi$, with a slope of $\frac{1}{2}\varphi$, intersects that curve. Furthermore, that mass is the only rational value that shows up for all positive powers of φ for any value of r_γ, m_γ, slope, or acceleration, and also v^2 in the space-time plane. Negative powers of φ do not produce physically *real* solutions to these equations, however, it is interesting to note that for φ^{-1}, the value for r_γ is the only other rational value for any powers of φ.

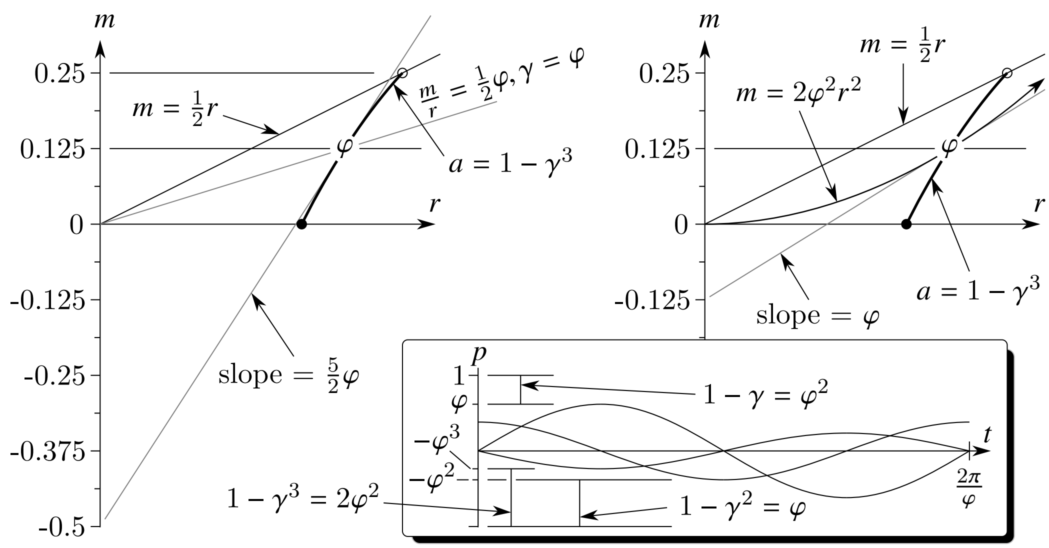

FIGURE 3. The golden ratio in a dimensional coordinate system. In the left pane, φ marks the spot (r_γ, m_γ) in the mass-space plane where the golden ratio, as the parameter γ, gives the arithmetic mean between the extremes of mass on the curve $a = 1 - \gamma^3$; the intersection of the line of constant $\gamma = \varphi$ with slope $\frac{1}{2}\varphi$, the curve $a = 1 - \gamma^3$, and its tangent line at that point with slope $\frac{5}{2}\varphi$, are also shown. In the right pane, the intersection of $a = 1 - \gamma^3$, the parabola of constant acceleration $2\varphi^2$, and its tangent line of slope φ at the point given by $\gamma = \varphi$ are shown. The inset shows relations for φ in the space-time plane when $\gamma = \varphi$: The amplitudes of $p(t) = \gamma \sin \gamma t$, and its first two derivatives are reduced by φ^2, φ, and $2\varphi^2$ respectively; the period is $\frac{2\pi}{\varphi}$.

Upon closer examination, it appears that φ might also occupy that point 'c' which satisfies the Mean Value Theorem for derivatives, giving the average rate of change of the curve. Since the endpoints of the curve are $(\frac{1}{3}, 0)$ and $(\frac{1}{2}, \frac{1}{4})$, the slope of the secant line is $\frac{3}{2}$. Using the parametric equations (3.10) and (3.11), we find the derivative is $\frac{5}{2}\varphi \approx 1.545$, not quite $\frac{3}{2}$. However, no other power of φ showed any expressible multiple of φ^n for the slope of the line tangent to (3.14). Finding the equation for that tangent line gives

$$m = \frac{5}{2}\varphi r - \frac{1}{2}, \tag{4.6}$$

an incredibly rational m-intercept for random placement. By contrast, the point 'c' which has a derivative equal to $\frac{3}{2}$, has a tangent line with m-intercept of ≈ -0.48157. Beginning at the lower end of (3.14), we find that for $r = \frac{1}{3}$, $\frac{1}{4\varphi}$, and $\lim_{r\to 1/2^-}$, the slopes of the tangent lines are 2, $\frac{5}{2}\varphi$, and 1 respectively, while the m-intercepts are $-\frac{2}{3}$, $-\frac{1}{2}$, and $-\frac{1}{4}$.

The transition from (3.14) to $m = ar^2$ is smooth and continuous when $a = 1$, as discussed in section 3. At $a = 1$, $m = ar^2$ has a derivative equal to one, and its tangent line has an m-intercept of $-\frac{1}{4}$; the same tangent line with slope given at that point by $\lim_{r\to 1/2^-}$ of the derivative of (3.14). When $\gamma = \varphi$, $a = 2\varphi^2$: for $m = ar^2$, the slope of the tangent line at $\gamma = \varphi$ is φ, and its m-intercept is $-\frac{1}{8}$. When $a = 0$, $\gamma = 1$, and the slope and m-intercept are both zero, and as such, whenever $-\frac{2}{3} \le m \le 0$ an m-intercept is defined by a tangent line to either (3.14), or $m = ar^2$. Furthermore, for $m = ar^2$, the m-intercept of the tangent lines is always the negative of the mass intersected by (3.14), and thus, the tangent line is bisected between the mass axis and (3.14) by the space axis.

These tangent lines also have the property that they are always of the form

$$m = (2ar_\gamma)r - m_\gamma, \tag{4.7}$$

such that the slope gives the right-hand term under the radical in (3.18), and the m-intercept gives the negative of the mass. Having noted that the slope gives the right-hand term under the radical, we set it equal to $2\frac{m}{r}$ and looking back at (4.2) through (4.5), we realize that for the sine wave with equation $p(t) = \gamma \sin \gamma t$, with period $\frac{2\pi}{\varphi^n}$ when $\gamma = \varphi^n$, the slope of these lines tangent to $m = ar^2$ is of the form

$$2\frac{m}{r} = \varphi^n L_n \tag{4.8}$$

for $n = 1, 3, 5, \ldots$, and

$$2\frac{m}{r} = \varphi^n F_n \sqrt{5} \tag{4.9}$$

for $n = 2, 4, 6, \ldots$, which is recognizable from (1.8) and general relativity.

5. Further Considerations

This is not the first appearance of the golden ratio in general relativity. In May of 2017, The *Quarterly* published Bryant and Hobill's *Golden-Ratio-Based Rectangular Tilings* [1], a work inspired by the discovery of the golden ratio in time evolved studies of the cosmological singularity based on Einstein's equations of general relativity [2]. However, we could find no connection between that appearance and what has been reported here: in [1] the tiling gives slopes that are φ^{-n}, reciprocal slopes of those discussed here, and in [2] the rectangles are time dependent. This occurrence of φ is characterized by its time independence: the mass-space plane is perpendicular to the time-axis; the time-axis here does not represent an evolution in time. Instead, the sine wave presented here represents all possible kinematic states with no respect to order, its dilation leads us to the conclusion $a = 1 - \gamma^3$. In the mass-space plane, that curve demonstrates a multitude of relations to φ when the parameter γ is equal to φ^n.

Most significantly we found that the golden ratio gives the arithmetic mean between the extremes of mass on the curve $a = 1 - \gamma^3$ in the mass-space plane. At that point the slope of a line of constant γ is $\frac{1}{2}\varphi$, the gravitational acceleration is $2\varphi^2$, changing at a rate of φ, and the slope of the line tangent to $a = 1 - \gamma^3$ is $\frac{5}{2}\varphi$. Similar relations of φ^n have been found for all of these metrics except the tangent line to $a = 1 - \gamma^3$. The amplitude and period of the sine wave in the space-time plane are also functions of φ^n under these conditions.

Heyrovska [6] has shown that electrostatic forces between positive and negative charges give rise to φ in atomic dimensions such as the ratios of cationic, anionic, and covalent radii. Livio [8] discusses many appearances of φ in nature, from the ridges in quasicrystals being in golden ratio, to Fibonacci based phyllotaxis in plants, and Bryant and Hobill have found φ in the ratio of the lengths of *epochs* during the formation of the universe. These occurrences of φ in nature consistently signal some form of equillibrium in golden ratio: interatomic distances, quasicrystal spacing, leaf rotation, and even the length of time between critical stress events in the formation of the universe. However, no such ratio has been noted here, in fact the golden ratio even gives a point of *bi*section rather than *golden* section here, making this occurrence somewhat unique.

6. Acknowledgments

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Physics Phor Phun Tutoring, 1297 Santa Rita Rd., Templeton, CA, 93465, USA
E-mail address: `rhanush@physicsphorphun.org`